Schwinger terms of the commutator of two interacting currents in the 1+1 dimensions

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Abstract

We calculate the equal-time commutator of two fermionic currents within the framework of the 1+1 dimensional *fully* quantized theory, describing the interaction of massive fermions with a massive vector boson. It is shown that the interaction does not change the result obtained within the theory of free fermions.

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I. INTRODUCTION

When quantizing the fermionic currents there occur additional terms in the equal-time (E.T.) commutators, so called Schwinger terms. They can be determined by different methods, for instance perturbatively, see e.g. [1], [2], [3], [4] or cohomologically [5], [6], and they have an important impact on the theory [1].

Schwinger terms are also closely related to the anomalies of QFT (for an overview see [7]). Whereas anomalies do not get altered by considering *quantized* gauge fields (due to the Adler-Bardeen theorem [8]) this is less clear for the Schwinger terms (ST). Therefore it is our aim to investigate ST in a *full* quantized theory. We will work in a 1+1 dimensional QFT describing the interaction of fermions with a massive vector field. There all calculations can be performed explicitly and it is a natural continuation of the work [9], where the case of the free fermions was discussed.

The fact that *all* fields are quantized distinguishes this work from others where the similar calculations were done for the theories describing fermions interacting with *external* fields (see e.g. [10], [11], [12] and the references given there).

The paper is organized as follows. In Section 2 we start with the definition of the interacting current J^{μ}_{int} [1] introduced by Bogoliubov [13] in the framework of the formalism of Epstein and Glaser [14] which is explained in detail and extensively used in the book of Scharf [15]. Using this definition we derive the explicit form of J^{μ}_{int} in the considered two dimensional field theory model. The calculation of the commutator is done in Section 3. In the Appendix we rigorously show that in the case of our model, the formalism of Epstein and Glaser is equivalent to the ordinary one using the $\mathcal{T}$-product (time-ordered product).

II. DEFINITION OF THE INTERACTING CURRENT

Following Bogoliubov [13] and Scharf [15] we define

$$J^{\mu}(x) \equiv \mathcal{S}^{-1}(g)\, \frac{\delta \mathcal{S}(g)}{i\delta g_{\mu}(x)}\bigg|_{g_{\mu}=0}, \tag{2.1}$$

where the S-matrix $\mathcal{S}$ and its inverse $\mathcal{S}^{-1}$ are expressed in perturbative form as

$$\begin{aligned}\mathcal{S}(g) &\equiv \mathbf{1} + \sum_{n=1}^{\infty}\sum_{i=0}^{n} \frac{e^{i}}{i!\,(n-i)!} \times \\ &\quad \times \int T_{n}^{\mu_1,\dots,\mu_{n-i}}(x_1,\dots,x_n) g_{\mu_1}(x_{i+1})\dots g_{\mu_{n-i}}(x_n)\, d^2x_1 \dots d^2x_n \qquad (2.2) \\ &\equiv \mathbf{1} + T, \qquad (2.3)\end{aligned}$$

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[1] Label *int* is to emphasize that the operator is *not* a composite operator.

$$\mathcal{S}^{-1}(g) \equiv \mathbf{1} + \sum_{n=1}^{\infty} \sum_{i=0}^{n} \frac{e^i}{i!\,(n-i)!} \times$$
$$\times \int \tilde{T}_n^{\mu_1,\ldots,\mu_{n-i}}(x_1,\ldots,x_n) g_{\mu_1}(x_{i+1}) \ldots g_{\mu_{n-i}}(x_n)\, d^2x_1 \ldots d^2x_n, \tag{2.4}$$

where e is the coupling constant of the interaction between fermions and bosons and $g_{\mu_i}(x)$ is a c-number test functions from $\mathcal{S}(\mathbb{R}^2)$ (Schwartz space). Properties of $T_n^{\mu_1,\ldots,\mu_{n-i}}(x_1,\ldots,x_n)$ are fixed by the required properties of $\mathcal{S}$ (see [15]).

From the equation

$$\mathcal{S}(g)^{-1} = (\mathbf{1} + T)^{-1} = \mathbf{1} + \sum_{r=1}^{\infty} (-T)^r \tag{2.5}$$

we get

$$\tilde{T}_n(X) = \sum_{r=1}^{n} (-)^r \sum_{P_r} T_{n_1}(X_1) \ldots T_{n_r}(X_r), \tag{2.6}$$

where the second sum runs over all partitions P_r of $X = \{x_1, x_2, \ldots, x_n\}$ into r disjoint non-empty subsets.

For the meaningful definition of our field theory model we have to define the first order terms of (2.2)

$$\mathcal{S}^{(1)}(g) \equiv i \int \{eT_1(x) + g_\mu(x) T_1^\mu(x)\}\, d^2x, \tag{2.7}$$

where

$$T_1(x) = :\overline{\psi}(x) \not{A} \psi(x):, \tag{2.8}$$
$$T_1^\mu(x) = :\overline{\psi}(x)\gamma^\mu\psi(x): . \tag{2.9}$$

The fields $\psi(x)$ and $\overline{\psi}(x)$ represent both fermion and antifermion and the A_μ is a vector boson field. All fields appearing in (2.8) and (2.9) are *free* since we work with a perturbation expansion. The masses of the particles we denote

$$m_{\psi(\overline{\psi})} = m,$$
$$m_A = \mu.$$

Note that we use the model containing a massive vector field in order to avoid problems with infrared singularities.

Using (2.1) and (2.2) we derive

$$J^\mu(x) = J^\mu_{free}(x) + \frac{1}{i} \sum_{n=1}^{\infty} \frac{1}{n!} \int A^\mu_{n+1}(x_1,\ldots,x_n;x)\, d^2x_1 \ldots d^2x_n, \tag{2.10}$$

where

$$J^{\mu}_{free}(x) =: \overline{\psi}(x)\gamma^{\mu}\psi(x) : \tag{2.11}$$

and A^{μ}_{n+1} is the so-called advanced $(n+1)$-point function

$$A^{\mu}_{n+1}(x_1,\ldots,x_n;x) = \sum_{P_2^0} \tilde{T}_m\left(X/Y\right) T^{\mu}_{n-m}\left(Y,x\right), \tag{2.12}$$

where $\sum_{P_2^0}$ means the summation over all partitions of the set X including the empty subset $X/Y = \emptyset$. The label *advanced* means that the support of A^{μ}_{n+1} is

$$\text{supp } A^{\mu}_{n+1}(x_1,x_2,\ldots,x_n;x) \subseteq \Gamma^{-}_{n+1}\left(x\right),$$

where

$$\Gamma^{-}_{n+1}\left(x\right) \equiv \left\{ \{x_i\}^n_{i=1} | \left(x_i - x\right)^2 \geq 0, x^0_i \leq x^0 \right\},$$

i.e. the A^{μ}_{n+1} vanish if an arbitrary x^0_i is greater then x^0.

For the reason which will become clear later we rewrite (2.12) in the form

$$A^{\mu}_{n+1}(x_1,\ldots,x_n;x) = \sum_{\Pi} \theta\left(x,x_{i_1},\ldots,x_{i_n}\right) C_n\left(x,x_{i_1},x_{i_2},\ldots,x_{i_n}\right), \tag{2.13}$$

where

$$\begin{aligned} C_n\left(x,x_{i_1},x_{i_2},\ldots,x_{i_n}\right) &= [\ldots[[T_1{}^{\mu}\left(x\right),T_1\left(x_{i_1}\right)],T_1\left(x_{i_2}\right)]\ldots,T_1\left(x_{i_n}\right)] \\ \theta\left(x,x_{i_1},\ldots,x_{i_n}\right) &= \theta\left(x^0 - x^0_{i_1}\right)\theta\left(x^0_{i_1} - x^0_{i_2}\right)\ldots\theta\left(x^0_{i_{n-1}} - x^0_{i_n}\right) \end{aligned} \tag{2.14}$$

and the summation runs over all permutations of the elements of X. (The 'advancing' of the support is then evident.)

Combining (2.10) and (2.13) we get

$$J^{\mu}(x) = \frac{1}{i}\sum_{n=0}^{\infty} e^n J^{\mu}_n(x), \tag{2.15}$$

where

$$J^{\mu}_0(x) = iJ^{\mu}_{free}(x) = T_1{}^{\mu}\left(x\right) \tag{2.16}$$

$$J^{\mu}_n(x) = \int \theta\left(x,x_1,x_2,\ldots,x_n\right) C_n\left(x,x_1,x_2,\ldots,x_n\right)\ d^2x_1 d^2x_2\ldots d^2x_n. \tag{2.17}$$

Nevertheless the definition (2.1) has to be slightly modified if we (naturally) require that the vacuum expectation value of the interacting current $J^{\mu}(x)$ be equal to zero.

Our redefinition is then straightforward

$$\begin{aligned} J^{\mu}_{int}(x) \equiv&\ J^{\mu}(x) - \langle 0|\, J^{\mu}(x)\,|0\rangle \\ \equiv&\ \frac{1}{i}\sum_{n=0}^{\infty} e^n\left(J^{\mu}_n(x) - \langle 0|\, J^{\mu}_n(x)\,|0\rangle\right) \end{aligned} \tag{2.18}$$

III. COMMUTATOR OF INTERACTING CURRENTS

Now we are ready to calculate the commutator of two interacting currents. Using (2.18) we write

$$[J^\mu_{int}(x), J^\nu_{int}(y)]_{E.T.} = [J^\mu(x), J^\nu(y)]_{E.T.} = -\sum_{n=0}^{\infty} e^n \sum_{i=0}^{n} \left[J^\mu_i(x), J^\nu_{n-i}(y)\right] . \tag{3.1}$$

and according to formulae (2.17), (2.14) and due to the fact [16] that

$$\begin{aligned} & A^\mu_{n+1}(x_1, \ldots, x_n; x) - A^\nu_{n+1}(x_1, \ldots, x_n; y) = \\ & \quad = \sum_{i_1 \ldots i_n} \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} \left[A^\mu_{n+1}(x_{i_1}, \ldots, x_{i_k}; x), A^\nu_{n+1}(x_{i_{k+1}}, \ldots, x_{i_n}; y)\right] \end{aligned} \tag{3.2}$$

we finally get

$$\begin{aligned} & [J^\mu_{int}(x), J^\nu_{int}(y)]_{E.T.} = \\ & \quad = -\sum_{n=0}^{\infty} e^n \int \theta\,(x, x_1, x_2, \ldots, x_n) \times \\ & \quad\quad \times [\ldots[[{T_1}^\mu\,(x)\,, {T_1}^\nu\,(y)]_{E.T.}\,, T_1\,(x_1)]\ldots, T_1\,(x_n)]\;\; d^2x_1 d^2x_2 \ldots d^2x_n. \end{aligned} \tag{3.3}$$

However, it turns out that only the first term in the sum contributes. To see this, one has to realize that the *operator relation*

$$[{T_1}^\mu\,(x)\,, {T_1}^\nu\,(y)]_{E.T.} \sim \mathbf{1} \tag{3.4}$$

is valid. Indeed, using Wick theorem we express $[T^\mu_1\,(x)\,, T^\nu_1\,(y)]$ in terms of the normally ordered products

$$\begin{aligned} [T^\mu_1\,(x)\,, T^\nu_1\,(y)] = {} & i : \overline{\psi}(y)\gamma^\nu S\,(y-x)\,\gamma^\mu \psi(x) : - \,i : \overline{\psi}(x)\gamma^\mu S\,(x-y)\,\gamma^\nu \psi(x) : + \\ & + \operatorname{tr}\left\{S^{(-)}\,(x-y)\,\gamma^\nu S^{(+)}\,(y-x)\,\gamma^\mu - S^{(-)}\,(y-x)\,\gamma^\mu S^{(+)}\,(x-y)\,\gamma^\nu\right\} . \end{aligned} \tag{3.5}$$

In the equal-time limit we have

$$S\,(x-y)\mid_{E.T.} = \gamma^0 \delta\left(x^1 - y^1\right) \tag{3.6}$$

and because in the 1+1 dimensions the identity

$$\gamma^\mu \gamma^0 \gamma^\nu = \gamma^\nu \gamma^0 \gamma^\mu \tag{3.7}$$

holds, the relation (3.4) is proved[2].

Thus, we can conclude that no contribution from the interaction appears, i.e.

$$[J^\mu_{int}(x), J^\nu_{int}(y)]_{E.T.} = [J^\mu_0(x), J^\nu_0(y)]_{E.T.} \tag{3.9}$$

This is the main result of the article.

As the author has checked the same result can be obtained in the bosonization scheme [18] therefore a connection between this scheme and our approach might exist.

IV. CONCLUSION

We have calculated the commutator of interacting currents in the simple two-dimensional model describing the interaction of fermions with a massive vector field. We have shown that the interaction does not change the result obtained within the theory of free fermions. A similar result we also expect to occur in 3+1 dimensions, however, the same calculations cannot be carried out in the same way in the 3+1 dimensions because the equality

$$\sum_{P_2^0} \tilde{T}_m\left(X/Y\right) T^\mu_{n-m}\left(Y,x\right) = \sum_{\Pi} \theta\left(x, x_{\pi_1}, \ldots, x_{\pi_n}\right) C_n\left(x, x_{\pi_1}, \ldots, x_{\pi_n}\right) \tag{4.1}$$

is no longer valid.

Taking the limit $\mu \to 0$ for the vector field would lead to problems because of the infrared singularities. Nevertheless these singularities do not preclude the possibility to split the causal distributions using multiplication by theta-functions. Therefore one may guess that the relation (3.9) remains valid even in the limit $\mu \to 0$.

In view of the intimate connection between Schwinger terms and anomaly, our result naturally suggests another (open) question concerning its possible relation to the Adler-Bardeen theorem. However, a *general* proof that quantized gauge fields do not change the result of the external fields is still missing.

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[2]Here we are working with normal ordered operators, which are well-defined in Fock space. Therefore we can indeed write

$$i : \overline{\psi}(y)\gamma^\nu S\left(y-x\right)\gamma^\mu\psi(x) : -i : \overline{\psi}(x)\gamma^\mu S\left(x-y\right)\gamma^\nu\psi(x) : |_{E.T.} = 0, \tag{3.8}$$

i.e. RHS of (3.8) is *not* given as the difference of two infinities [17].

APPENDIX: THE TRANSITION STEP

To justify the transition from (2.12) to (2.13) we introduce the distribution D^μ_{n+1}

$$D^\mu_{n+1}(x_1,\ldots,x_n;x) \equiv R^\mu_{n+1}(x_1,\ldots,x_n;x) - A^\mu_{n+1}(x_1,\ldots,x_n;x), \tag{A.1}$$

where

$$R^\mu_{n+1}(x_1,\ldots,x_n;x) \equiv \sum_{P_2^0} T^\mu_{n-m}\left(Y,x\right)\tilde{T}_m\left(X/Y\right). \tag{A.2}$$

It is possible to show that

$$\text{supp } R^\mu_{n+1}(x_1,x_2,\ldots,x_n;x) \subseteq \Gamma^+_{n+1}\left(x\right), \tag{A.3}$$

where

$$\Gamma^+_{n+1}\left(x\right) \equiv \left\{ \left\{x_i\right\}_{i=1}^n \middle| \left(x_i - x\right)^2 \geq 0, x_i^0 \geq x^0 \right\} \tag{A.4}$$

and therefore D^μ_{n+1} has a causal support, i.e.

$$\text{supp } D^\mu_{n+1}\left(X,x\right) \subseteq \Gamma^+_{n+1}\left(x\right) \cup \Gamma^-_{n+1}\left(x\right). \tag{A.5}$$

It is clear that if D^μ_{n+1} is not singular then A^μ_{n+1} can be expressed as

$$A^\mu_{n+1}(x_1,\ldots,x_n;x) = -\prod_{i=1}^n \theta\left(x^0 - x_i^0\right) D^\mu_{n+1}(x_1,\ldots,x_n;x). \tag{A.6}$$

Further we introduce the 'truncated' distributions $A'^{\,\mu}_{n+1}$ and $R'^{\,\mu}_{n+1}$

$$A'^\mu_{n+1} \equiv \sum_{P_2} \tilde{T}_m\left(X/Y\right) T^\mu_{n-m}\left(Y,x\right), \tag{A.7}$$

$$R'^{\,\mu}_{n+1} \equiv \sum_{P_2} T^\mu_{n-m}\left(Y,x\right)\tilde{T}_m\left(X/Y\right), \tag{A.8}$$

where $\sum_{P_2}$ means summation over all partitions of the set X to non-empty subsets. Using (A.7) and (A.8) the distribution (A.1) can be express as

$$D^\mu_{n+1} = R'^{\,\mu}_{n+1} - A'^{\,\mu}_{n+1}. \tag{A.9}$$

There is one important difference between (A.1) and (A.9). The latter gives us the possibility to express D^μ_{n+1} in terms of the n-point function T^n.

Example:

$$D^\mu_2(x_1;x) = R'^\mu_2(x_1;x) - A'^\mu_2(x_1;x) \tag{A.10}$$

and if D^μ_2 is not singular then we can write

$$A_2^\mu(x_1;x) = -\theta\left(x^0 - x_1^0\right) D_2^\mu(x_1;x), \tag{A.11}$$

i.e. we split D_2^μ. Then using (A.11) we get

$$\begin{aligned} A_2^\mu(x_1;x) &= -\theta\left(x^0 - x_{\pi_1}^0\right)\left({R'_2}^{\mu}(x_1;x) - {A'_2}^{\mu}(x_1;x)\right) = \\ &= \theta\left(x^0 - x_{\pi_1}^0\right)\left[T_1^\mu(x), T_1(x_1)\right]. \end{aligned} \tag{A.12}$$

Furthermore according to the definitions (2.12), (A.7), (A.2) and (A.8)

$$T_n^\mu\left(x_1, \ldots, x_{n-1}, x\right) = R_n^\mu - {R'_n}^{\mu} = A_n^\mu - {A'_n}^{\mu} \tag{A.13}$$

and using repeatedly (A.6) we can finally express D_{n+1}^μ in the terms of the 1-point function $T_1^{(\mu)}$. In that way we get A_{n+1}^μ in (2.13) by combining (A.6) and the above procedure.

This routine is equivalent [16] to the application of following equality

$$T_{n+1}^\mu\left(x_1, \ldots, x_{n-1}, x\right) = \mathcal{T}\left(T_1^\mu(x)T_1(x_1)\ldots T_1(x_n)\right),$$

where

$$\mathcal{T}\left(T_1^\mu(x)T_1(x_1)\ldots T_1(x_n)\right) = \sum_{\Pi}\theta\left(x, x_{i_1}, \ldots, x_{i_n}\right) T_1^\mu(x)T_1(x_{i_1})\ldots T_1(x_{i_n}). \tag{A.14}$$

Example:

$$\begin{aligned} T_2^\mu\left(x_1, x\right) &= -\theta\left(x^0 - x_1^0\right) D_2^\mu - {A'_2}^{\mu} = \\ &= -\theta\left(x^0 - x_1^0\right) {R'_2}^{\mu} - \left(1 - \theta\left(x^0 - x_1^0\right)\right) {A'_2}^{\mu} = \\ &= \theta\left(x^0 - x_1^0\right) T_1^\mu(x)T_1(x_1) + \theta\left(x_1^0 - x^0\right) T_1^\mu(x)T_1(x_1) = \\ &= T\left(T_1^\mu(x)T(x_1)\right). \end{aligned} \tag{A.15}$$

However, as it was shown in [15] the splitting of an arbitrary distribution with the causal support to retarded and advanced part via multiplication by the combination of theta-functions, i.e. (A.6) is not generally a well defined procedure.

The reason why we can do it here is that we work in two dimensions. We show that the terms $T_{i_1}\ldots T_{i_k}, i_j \in \{1, \ldots, n\}$, $\sum_{j=1}^k i_j = n+1$,which are ‘sitting’ in D_{n+1}, are correctly defined and they have non singular behavior. The last enables their multiplication by the combination of the theta-functions.

Every term $T_{i_1}\ldots T_{i_k}$ is expressible as a sum of terms of the normally ordered operators (graphs) of the form

$$T_{i_1}\ldots T_{i_k} \sim \sum_k T_{n+1}^{g_k}(x_1, \ldots, x_n, x) \tag{A.16}$$

where

$$T_n^g(x_1, \ldots, x_n) =: \prod_{i=1}^{n_f}\overline{\psi}(x_{k_j}) t_g\left(x_1, \ldots, x_n\right)\prod_{i=1}^{n_f}\psi(x_{n_j}) :: \prod_{i=1}^{n_b} A\left(x_{m_j}\right): \tag{A.17}$$

and n_f is the number of external fermions (or antifermions), n_b the number of the external massive bosons and $t_g(x_{1,}\ldots,x_n)$ is c-number distribution.

In the dimension d the graph g (A.17) has the singular order

$$\omega(g) = n\left(\frac{d}{2}-2\right) + d - n_b\left(\frac{d}{2}-1\right) - n_f(d-1) \tag{A.18}$$

and for $d=2$ we get

$$\omega(g) = 2 - n - n_f. \tag{A.19}$$

We see that the problematic (singular) case $\omega(g) \geq 0$ can appear only for $n=2$. All higher-order graphs do not contain any singularity. Moreover the c-number distribution in the 2-point causal function has really $\omega(g) = -2$. This all means that all graphs (including their subgraphs) are not singular, the terms $T_{i_1}\ldots T_{i_k}$ are well-defined and can be multiplied by the combination of theta-functions.

Therefore the formula (2.13) is consistent with (2.12).

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